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Trajectory Optimization Based on Differential Inclusion

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A method for generating finite-dimensional approximations to the solutions of optimal control problems is introduced. By employing a description of the dynamical system in terms of its attainable sets in favor of using differential equations, the controls are completely eliminated from the system model. Besides reducing the dimensionality of the discretized problem compared to state-of-the-art collocation methods, this approach also alleviates the search for initial guesses from where standard gradient search methods are able to converge. The mechanics of the new method are illustrated on a simple double integrator problem. The performance of the new algorithm is demonstrated on a one-dimensional rocket ascent problem (Goddard Problem) in presence of a dynamic pressure constraint.

Introduction

The precise solution of an optimal control problem via Pontryagin's minimum principle involves the numerical treatment of nonlinear multipoint boundary value problems (BVPs). The structure of these BVPs depends on the sequence in which the optimal control switches between singular/nonsingular and constrained/unconstrained arcs and is not known to the analyst in advance. Additionally, these BVPs involve costate variables whose physical meaning offers little help in determining reasonable initial guesses from which gradient search methods can converge.

For these reasons, rapid trajectory generation is usually attempted by applying direct optimization techniques to some type of discretized problem formulation. This approach leads to the numerical task of solving nonlinear programming problems. The performance of the optimization algorithm involved and the precision of the obtained solutions depends strongly on the chosen problem discretization and on the dimension of the associated parameter space.

One discretization approach is based on collocation[1,2] as implemented in the optimal trajectories by implicit simulation[3,4] (OTIS) program. The algorithm introduced in the present paper is very similar in its structure to the OTIS approach. However, the derivation involves a representation of the dynamical system in differential inclusion format, where the right-hand side of the equations of motion is set-valued. The advantage of the new approach lies in the fact that it is completely void of controls. Besides the lower dimensionality of the discretized problem and the reduced number of initial guesses that need to be provided by the user, the absence of control variables seems to have a positive effect on the convergence behavior, especially for singular control problems.

Problem Formulation in Terms of Differential Equations

We consider optimal control problems of the following general form:

$$\min_{u \in (PWC[0,1])^m} \quad \Phi[x(0), x(1)] \tag{1}$$

subject to the equations of motion

$$\dot{x}(t) = f[x(t), u(t)] \qquad \forall\, t \in [0, 1] \tag{2}$$

the boundary conditions

$$\Psi[x(0), x(1)] = 0 \tag{3}$$

and the control constraints

$$g[x(t), u(t)] = 0 \qquad \forall\, t \in [0, 1] \tag{4}$$

$$h[x(t, u(t)] \le 0 \qquad \forall\, t \in [0, 1] \tag{5}$$

Here, $t \in \boldsymbol{R}$, $x(t) \in \boldsymbol{R}^n$, and $u(t) \in \boldsymbol{R}^m$ are normalized time, state vector, and control vector, respectively. The functions $\Phi : \boldsymbol{R}^{2n} \to \boldsymbol{R}$, $f : \boldsymbol{R}^{n+m} \to \boldsymbol{R}^n$, $\Psi : \boldsymbol{R}^{2n} \to \boldsymbol{R}^s$, $s \le 2n$, and $g : \boldsymbol{R}^{n+m} \to \boldsymbol{R}^{k_g}$, $h : \boldsymbol{R}^{n+m} \to \boldsymbol{R}^{k_h}$ are assumed to be sufficiently smooth with respect to their arguments of whatever order is required in this paper. $(PWC[0,1])^m$ denotes the set of all piecewise continuous functions that map the interval [0,1] into $\boldsymbol{R}^m$. It should be noted that also free final time problems can be transformed into the general form (1–5).

Hodograph

For fixed states x, the hodograph $S(x)$ is defined as the set of all possible state rates $\dot{x}$ that can be achieved by varying the controls within their allowed bounds (see Ref. 5). Given the state equations (2) and the control constraints (4) and (5), we can write

$$S(x) = \{\dot{x} \in \mathbf{R}^n | \dot{x} = f(x, u),\ u \in \Omega(x)\} \tag{6}$$

where $\Omega(x)$ is the set of all admissible controls $u \in \boldsymbol{R}^m$, i.e.,

$$\Omega(x) = \{u \in \mathbf{R}^m | g(x, u) = 0,\ h(x, u) \le 0\} \tag{7}$$

In the formulation (7), the controls $u \in \boldsymbol{R}^m$ can be regarded as a mere instrument for parameterizing the hodograph (6). In fact, the optimal state history and the optimal cost associated with problem (1–5) are not affected if the control vector $u \in \boldsymbol{R}^m$ and the control constraints (4) and (5) are replaced by any other set of variables to parameterize the admissible state rates as long as the resulting hodograph remains unchanged.

We now assume that there are smooth functions $p : \boldsymbol{R}^{2n} \to \boldsymbol{R}^{l_p}$, $q : \boldsymbol{R}^{2n} \to \boldsymbol{R}^{l_q}$, such that the hodograph $S(x)$ defined in Eqs. (6) and (7) can be rewritten as

$$S(x) = \{\dot{x} \in \mathbf{R}^n | p(\dot{x}, x) = 0,\ q(\dot{x}, x) \le 0\} \tag{8}$$

To guarantee the existence of such functions p and q, we may replace $S(x)$ by its convex hull.[2,7] This avoids difficulties of the nature that arise, for instance, when the hodograph is not a connected set,[5] i.e., consists of two or more nonempty, disjoint

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subsets. Replacing the hodograph by its convex hull is sometimes called relaxing the problem. If the solution to the relaxed problem has its state rates always operating in the domain of the original nonconvex hodograph, then obviously the problem relaxation did not have any effect on the solution. In this case, the solutions to the relaxed and the unrelaxed problem are the same. If the solution to the relaxed problem has state rates operating outside the original nonconvex hodograph, then this indicates chattering control[7] and a solution to the unrelaxed problem does not exist. Hence, no information is lost by replacing the hodograph of the original problem (1–5) through its convex hull. For the remainder of the paper we assume, without loss of generality, that the hodograph defined in Eqs. (6) and (7) is convex.

The aim of the subsequent sections is to exploit Eq. (8) to introduce a problem formulation that is completely devoid of controls, employing only the information condensed in the hodograph. Using the concept of differential inclusions,[8,9] the evolution of the dynamical system can be described completely in terms of states and their sets of attainability.

Sets of Attainability

Given a starting time t_0, an initial state $x(t_0) = x_0$, and a final time t_1, the set of attainability $K(t_0, x_0; t_1)$ is defined to consist of all points $x \in \boldsymbol{R}^n$ to which the state vector $x(t)$ can be steered at time t_1 through an admissible control $u(t)_{t \in [t_0, t_1]}$. Here, an admissible control is any function of time $u(t) \in PWC[0,1]$ restricted to the subinterval $[t_0, t_1]$ that satisfies the control constraints (4) and (5). The dependence of the attainable set on the right-hand side of the state equations f, and on the control constraints g, h, is suppressed in the nomenclature $K(t_0, x_0, t_1)$.

Now, let Δt be a small time step initiated at time t_0 and let $t_1 = t_0 + \Delta t$. Then, a first-order approximation $\tilde{K}(t_0, x_0; t_1)$ to the attainable set $K(t_0, x_0; t_1)$ can be given by

$$\tilde{K}(t_0, x_0; t_1) = \{x \in \mathbf{R}^n | x = x_0 + \Delta t \cdot S(x_0)\} \tag{9}$$

Here, we used the simplifying notation

$$\{\Delta t \cdot S(x_0)\} := \{\Delta t \cdot \dot{x} | \dot{x} \in S(x_0)\} \tag{10}$$

and S denotes the hodograph defined in Eq. (8). The notion of first-order approximation is understood in the sense that for fixed $\Delta t_{\max}$, there is a real number $M(\Delta t_{\max}) > 0$ such that for all $0 < \Delta t \le \Delta t_{\max}$ each element of the attainable set $K(t_0, x_0, t_1)$ can be approximated to first order by some element in $\tilde{K}(t_0, x_0, t_1)$. This means, for each element $x_1 \in K(t_0, x_0, t_1)$ there is an element $\tilde{x}_1 \in \tilde{K}(t_0, x_0, t_1)$ such that $\|x_1 - \tilde{x}_1\|_2 \le M \cdot \Delta t^2$.

New Numerical Approach

Let $0 = t_0, t_1, \ldots, t_{N-1}, t_N = 1$ be a user-chosen subdivision of the interval [0,1]. For simplicity, let the nodes be distributed equidistantly, i.e.,

$$t_i = i/N, \qquad i = 0, \cdots, N \tag{11}$$

A generalization to arbitrary subdivisions is straightforward. Then, let $x_0, x_1, \ldots, x_N \in \boldsymbol{R}^n$ be approximations to the states at time $t_0, \ldots, t_N$, respectively, and define

$$X = [x_0 | x_1 | \cdots | x_N] \in \mathbf{R}^{(N+1) \cdot n} \tag{12}$$

Problem (1–4) can now be rewritten in the form

$$\min_{X \in \mathbf{R}^{(N+1) \cdot n}} \Phi(x_0, x_N) \tag{13}$$

subject to the constraints

$$\Psi(x_0, x_N) = 0 \tag{14}$$

and

$$x_{i+1} \in K(t_i, x_i, t_{i+1}), \qquad i = 0, \cdots, N-1 \tag{15}$$

By substituting the attainable set $K(t_i, x_i, t_{i+1})$ by its first-order approximation $\tilde{K}(t_i, x_i, t_{i+1})$ as given by Eqs. (9) and (15) leads to

$$x_{i+1} \in \{x \in \mathbf{R}^n | x = x_i + \Delta t \cdot S(x_i)\} \tag{16}$$

Employing the assumption that the hodograph $S(x)$ defined in Eqs. (6) and (7) can be expressed in the form (8), it is clear that Eq. (16), after approximating x_i by

$$\dot{\bar{x}}_i = \frac{x_{i+1} - x_i}{\Delta t} \tag{17}$$

can be substituted equivalently by the conditions

$$\left.\begin{aligned} p(\dot{\bar{x}}_i, x_i) &= 0 \\ q(\dot{\bar{x}}_i, x_i) &\le 0 \end{aligned}\right\} i = 0, \cdots, N-1 \tag{18}$$

In summary, it is proposed to obtain the approximate values of the optimal state vectors x_i at times t_i, $i = 0, \ldots, N$, by solving the nonlinear parameter optimization problem (13) subject to the constraints (14) and (18). The values of the state variables x_i at the node points t_i, $i = 0, \ldots, N$ are the parameters that have to be found such that the performance index (13) is minimized. No controls are involved explicitly. The minimization is subject to the boundary conditions (14) and the interior constraints (18). For every state x_i that is picked at time t_i, the set of attainability $K(t_i, x_i; t)$, i.e., the set of coordinates to which the state vector can be steered after time t_i, keeps ballooning as time progresses. The conditions (18) enforce that the state x_{i+1} lies within the (first-order approximation to) attainable set $K(t_i, x_i; t_{i+1})$ (see Fig. 1). Hence, conditions (18) represent a numerical implementation of the differential inclusion concept.[8,9]

Extension 1: Higher Order Approximation

A more precise discretization can be obtained if condition (16) is substituted by

$$x_{i+1} \in \{x \in \mathbf{R}^n | x = x_i + \Delta t \cdot S(\bar{x}_i)\} \tag{19}$$

where

$$\bar{x}_i := \frac{x_i + x_{i+1}}{2} \tag{20}$$

In complete analogy to Eq. (18) this leads to

$$\left.\begin{aligned} p(\dot{\bar{x}}_i, \bar{x}_i) &= 0 \\ q(\dot{\bar{x}}_i, \bar{x}m_i) &\le 0 \end{aligned}\right\} i = 0, \cdots, N-1 \tag{21}$$

A derivation of the order of this approximation is not given in the present paper.

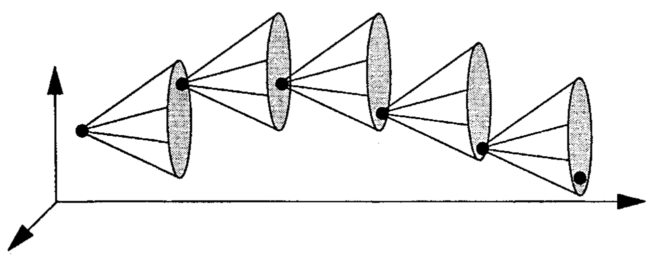

Fig. 1 Schematic description of the differential inclusion approach.

Extension 2: Nonequidistant Subdivision

For practical applications it may be useful to place the nodes t_i nonequidistantly rather than as defined in Eq. (11). For instance, if preliminary results obtained by equidistant node placement suggest rapid state transitions in some domain of the time interval, then it is advisable to rerun the problem with the same number of nodes, placed more densely in the areas of rapid state transitions and more scarcely in areas with sluggish state rates. Without increasing the number of parameters used to represent the discretized problem, this can significantly improve the precision of the result.

Formally, nonequidistant node placement does not complicate the discretized problem formulation (13), (14), and (16).

Extension 3: State Constraints

State equality or inequality constraints of the general form

$$v(x) = 0, \qquad w(x) \le 0 \tag{22}$$

usually represent a significant complication of the optimal control problem (1–5). For the discretization proposed in this paper, constraints (22) can be easily handled. By enforcing pointwise satisfaction of Eq. (22) we obtain the additional conditions

$$\left.\begin{aligned} v(x_i) &= 0 \\ w(x_i) &\le 0 \end{aligned}\right\} i = 0, \ldots, N \tag{23}$$

Then, the suboptimal solution to problems (1–5), and (22) is obtained by simply adding constraints (23) to the nonlinear programming problem of (13), (14), and (18). In contrast to indirect optimal control approaches based on the minimum principle[7,10,11] the user need not provide any guesses of the optimal switching structure.

Extension 4: Analytical Derivations

The numerical approach proposed in the present paper does not require explicit integration of the equations of motion. Instead, a number of equality and inequality constraints is imposed on any pair of neighboring states. As a consequence, the partial derivatives of the cost gradients and the constraint gradients, required by any Newton type method to solve the Kun-Tucker conditions associated with problems (13), (14), (18), and (23), can be calculated analytically as long as the functional dependencies of Φ, Ψ, p, p, q, v, and w on their arguments are known. With this rather easy access to analytical partial derivatives of the cost and constraint functions associated with the discretized optimization problem, the expensive evaluation of partial derivatives through finite differences can be eliminated. It can also be expected that analytical differentiation provides higher precision, which may be a deciding factor in case of a badly conditioned problem.

Extension 5: Explicit Time Dependence

In the problem formulation (1–5) and (22), no explicit time dependence of the describing functions Φ, Ψ, f, g, h, v, and w is assumed. This does not represent a serious restriction. Explicit dependence of the right-hand side of the state equations f on time t, for example, can be transformed away by introducing the additional state equation and initial condition

$$\dot{\tau} = 1, \qquad \tau(0) = 0 \tag{24}$$

thus providing a state carrying the value of the current time. Variable final time problems can be dealt with by introducing the additional state T through

$$\dot{T} = 0 \tag{25}$$

and multiplying the right-hand side f associated with all other states with T.

These techniques are very common and well known, and they can be applied to transform general optimal control problems to problems of the form of Eqs. (1–5) and (22). However, for each additional state introduced, the number of parameters in the discretized formulation (13), (14), (18), and (23), increases by $N + 1$, where N is the user chosen number of discretization nodes. However, in this discretized formulation it is not necessary to explicitly carry along conditions of the type (24) and (25). Through analytical integration, conditions (24) can be eliminated completely, and, in case of condition (25), the unknown constant of integration gives rise to a single scalar parameter that has to be added to the optimization parameters (12). In general, to keep the number of parameters in the nonlinear programming problem (13), (14), (18), and (23) small, it is advisable to customize the numerical approach by using analytical integration whenever possible. The implications for the analysis outlined in the preceeding sections are rather straightforward and the numerical benefits may be worth the extra effort.

Example 1: Double Integrator Problem

As an academic example to demonstrate the general procedures required by the new approach, we consider the following problem:

$$\min_{u \in PWC[0,1]} \; -x(1) \tag{26}$$

subject to the equations of motion

$$\dot{x} = v, \qquad \dot{v} = u \tag{27}$$

the initial and final conditions

$$x(0) = 0, \qquad v(0) = 0, \qquad v(1) = 0 \tag{28}$$

and the control constraints

$$-1 \le u \le 1 \tag{29}$$

The optimal control solution to this problem is of a bang-bang type. The associated state time histories are given in Fig. 2.

To apply the proposed algorithm, we first choose an integer N and define $N + 1$ (equidistantly placed) nodes

$$t_i = i/N, \qquad i = 0, \cdots, N \tag{30}$$

Then the values of the states $[x_i, v_i]^T$ at the nodes t_i, $i = 0, \ldots, N$, are obtained from solving the constrained parameter optimization problem

$$\min_{[(x_i, v_i)_{i=0,\cdots,N}]} \; -x(N) \tag{31}$$

subject to the constraints

$$x_0 = 0, \qquad v_0 = 0, \qquad v_N = 0 \tag{32}$$

and

$$\left.\begin{aligned} \dot{\bar{\mathrm{x}}}_i - \bar{v}_i &= 0 \\ \dot{\bar{v}}_i - 1 - &\le 0 \\ -\dot{\bar{v}}_i - 1 - &\le 0 \end{aligned}\right\} i = 0, \cdots, N - 1 \tag{33}$$

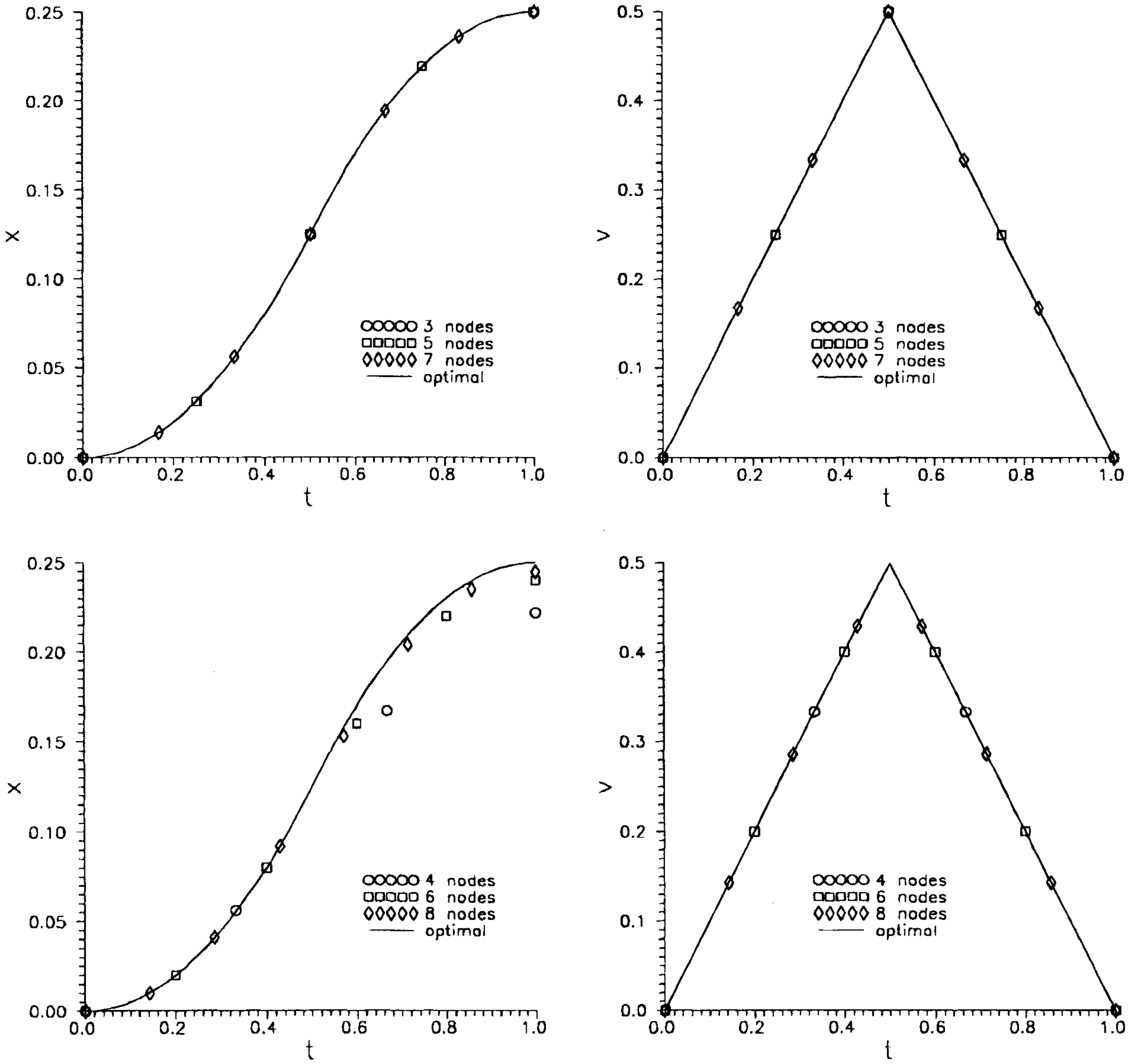

Fig. 2 Example 1: states x and v vs time t for several different node densities.

Here, $\bar{x}_i, \bar{v}_i, \dot{\bar{x}}_i, \dot{\bar{v}}_i$, $i = 0, \ldots, N-1$ are defined by

$$\bar{x}_i = \frac{x_{i+1} + x_i}{2}, \qquad \dot{\bar{x}}_i = \frac{x_{i+1} - x_i}{\Delta t}$$
$$\bar{v}_i = \frac{v_{i+1} + v_i}{2}, \qquad \dot{\bar{v}}_i = \frac{v_{i+1} - v_i}{\Delta t} \tag{34}$$

with

$$\Delta t = 1/N \tag{35}$$

and give approximate values for states and state rates in between nodes. Conditions (32) represent the initial/final conditions (28) in the discretized form, and conditions (33) replace the description (27) and (29) of the underlying dynamical system. For the derivation of conditions (33), the hodograph defined in Eq. (6) and (7)

$$S(x, v) = \{[\dot{x}, \dot{v}] \in \mathbf{R}^2 | \dot{x} = v, \dot{v} = u, u \in \Omega\} \tag{36}$$

$$\Omega(x, v) = \{u \in \mathbf{R} | -1 \leq u \leq 1\} \tag{37}$$

is rewritten in the general form (8)

$$S(x, v) = \{[\dot{x}, \dot{v}] \in \mathbf{R}^2 | \dot{x} - v = 0, \dot{v} - 1 \leq 0, -\dot{v} - 1 \leq 0,\} \tag{38}$$

Then the conditions $\dot{x} - v = 0$, $\dot{v} - 1 \leq 0$, $-\dot{v} - 1 \leq 0$ in Eq. (38) are, loosely speaking, evaluated in between subsequent nodes to yield Eq. (33).

It is well known that the optimal solution to problem (26–29) is of a bang-bang nature with $u(t) \equiv +1$ for $0 \leq t \leq 0.5$ and $u(t) \equiv -1$ for $0.5 \leq t \leq 1$ (see Fig. 2). From the linearity of the state equations (27), it follows that the discretized solution, i.e., the solution to Eqs. (31–35), will match the optimal solution perfectly as long as the control associated with the optimal solution is constant throughout each discretization interval. Noting that the optimal solution has a switching point only at time $t = 0.5$ and is identically constant elsewhere, it is clear that the discretized solution is identical to the optimal solution if and only if N is an even integer (odd number of nodes). This observation is also confirmed by the numerical results (see Fig. 2). All numerical results were obtained by employing the nonlinear programming code NPSOL[12,13] to solve the nonlinear programming problem (31–33). Providing initial guesses for the parameters $(x_i, v_i)_{i=0,\ldots,N}$, from where NPSOL was able to converge was not difficult. Convergence was always achieved in no more than three iterations even if the initial guesses were chosen many orders of magnitude off the respective optimal values.

Example 2: One-Dimensional Rocket Ascent (Goddard Problem)

As a nontrivial problem to demonstrate the performance of the new algorithm, we consider the problem of maximizing the final altitude for a sounding rocket ascending vertically under the influence of atmospheric drag and an inverse-square gravitational field. The states are radial distance r, velocity v, and mass m. The thrust magnitude T is the only control and is subject to fixed bounds $0 \leq T \leq T_{\max}$ (control constraints) and a dynamic pressure limit $q \leq q_{\max}$ (state constraint).

In nondimensional form the problem is given as follows:

$$\min - \tau(t_f) \tag{39}$$

subject to the equations of motion

$$\begin{aligned} \dot{r} &= v \\ \dot{v} &= (T - D/m) - (1/r^2) \\ \dot{m} &= -(T/c) \end{aligned} \tag{40}$$

the control constraint

$$T \in [0, T_{max}] \tag{41}$$

the boundary conditions

$$r(0) = 1 \tag{42a}$$

$$v(0) = 0 \tag{42b}$$

$$m(0) = 1 \tag{42c}$$

$$r(t_f) \text{ to be maximized} \tag{42d}$$

$$v(t_f) \text{ free} \tag{42e}$$

$$m(t_f) = m_f \tag{42f}$$

and the state inequality constraint

$$v - \sqrt{\frac{2q_{\max}}{\rho_0 e^{\beta(1-r)}}} \leq 0 \tag{43}$$

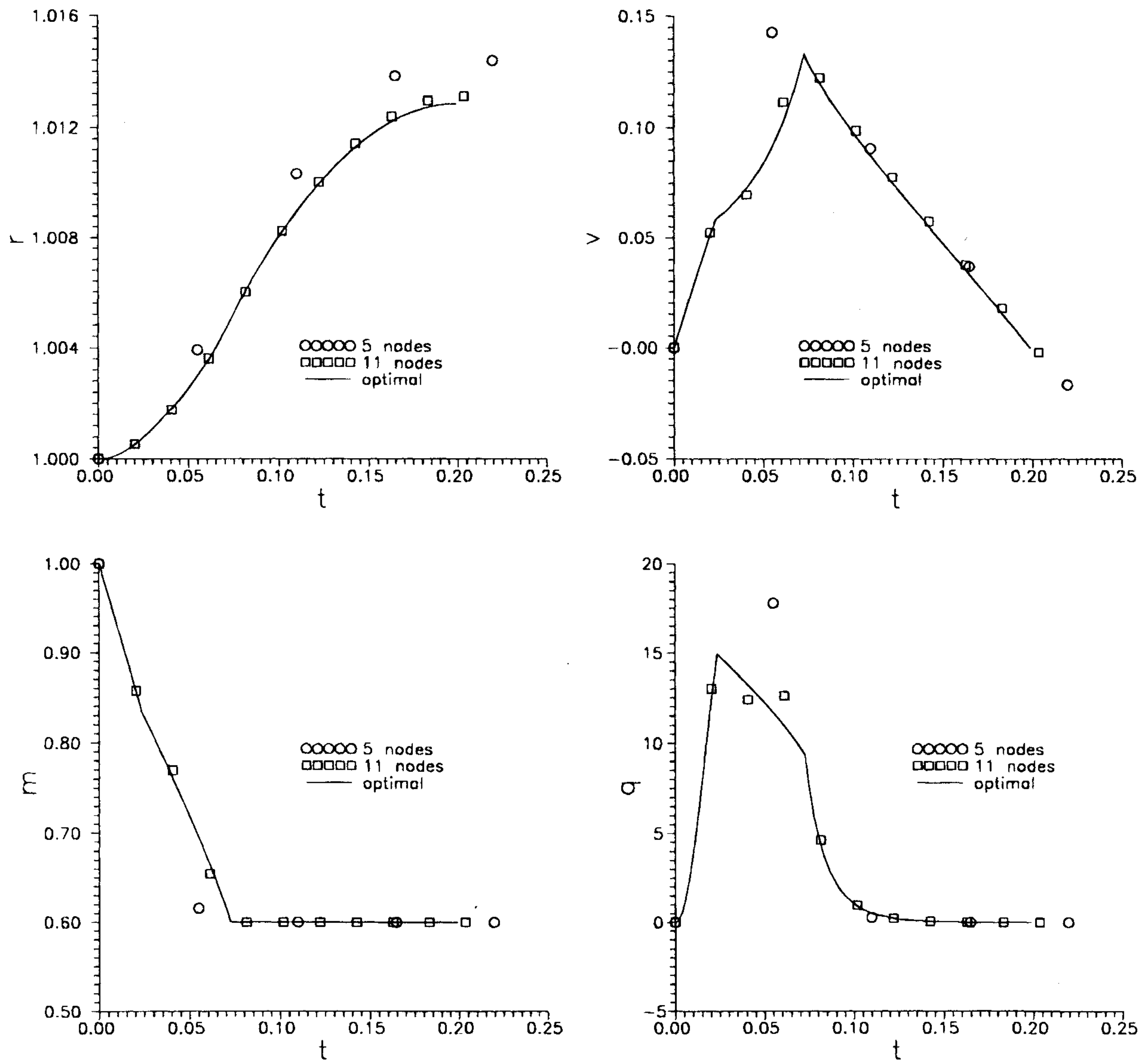

Fig. 3 Example 2: time histories for states r, v, m, and dynamic pressure q for $q_{\max} = \infty$ and various node densities.

With dynamic pressure q and air density ρ given by

$$q = \tfrac{1}{2}\rho v^2$$
$$\rho = \rho_0 e^{\beta(1-r)}$$

respectively, it is clear that the "speed limit" Eq. (43) is equivalent to a dynamic pressure limit $q - q_{max} \le 0$. The aerodynamic drag D is given by

$$D = qC_DA$$

The constants C_D, A, ρ_0, and β denote drag coefficient, cross-sectional area, air density at ground level, and exponential decay rate of air density with altitude, respectively. The constants c, T_{max}, and m_f used in Eqs. (40–42) denote the exhaust velocity, the maximum available thrust, and the final mass of the vehicle (after all of the fuel is burned), respectively. The nondimensional values used for numerical calculations are as follows:

$$C_D = 0.05$$
$$\rho_0 \cdot A = 12{,}400$$
$$\beta = 500 \qquad (44)$$
$$c = 0.5$$
$$T_{max} = 3.5$$
$$m_f = 0.6$$

A precise treatment of the problem employing optimal control techniques is presented in Ref. 14. For $q_{max} = \infty$, the time histories of the optimal states and the associated dynamic pressure are given in Fig. 3. For $q_{max} = 10$, the results are shown in Figs. 4 and 5. It should be noted that the solutions for both, $q_{max} = \infty$ and $q_{max} = 10$, involve singular control subarcs (compare Ref. 14).

To apply the numerical techniques introduced in the previous sections, we first choose an integer N and define $N + 1$ (equidistantly placed) nodes

$$t_i = i/N, \qquad i = 0, \cdots, N \qquad (45)$$

Then the values of the states $[r_i, v_i, m_i]^T$ at the nodes t_i, $i = 0, \ldots, N$, are obtained from solving the constrained parameter optimization problem

$$\min_{[(r_i,\, v_i,\, m_i)_{i=0,\cdots,N},\, t_f]} \; -r_N \qquad (46)$$

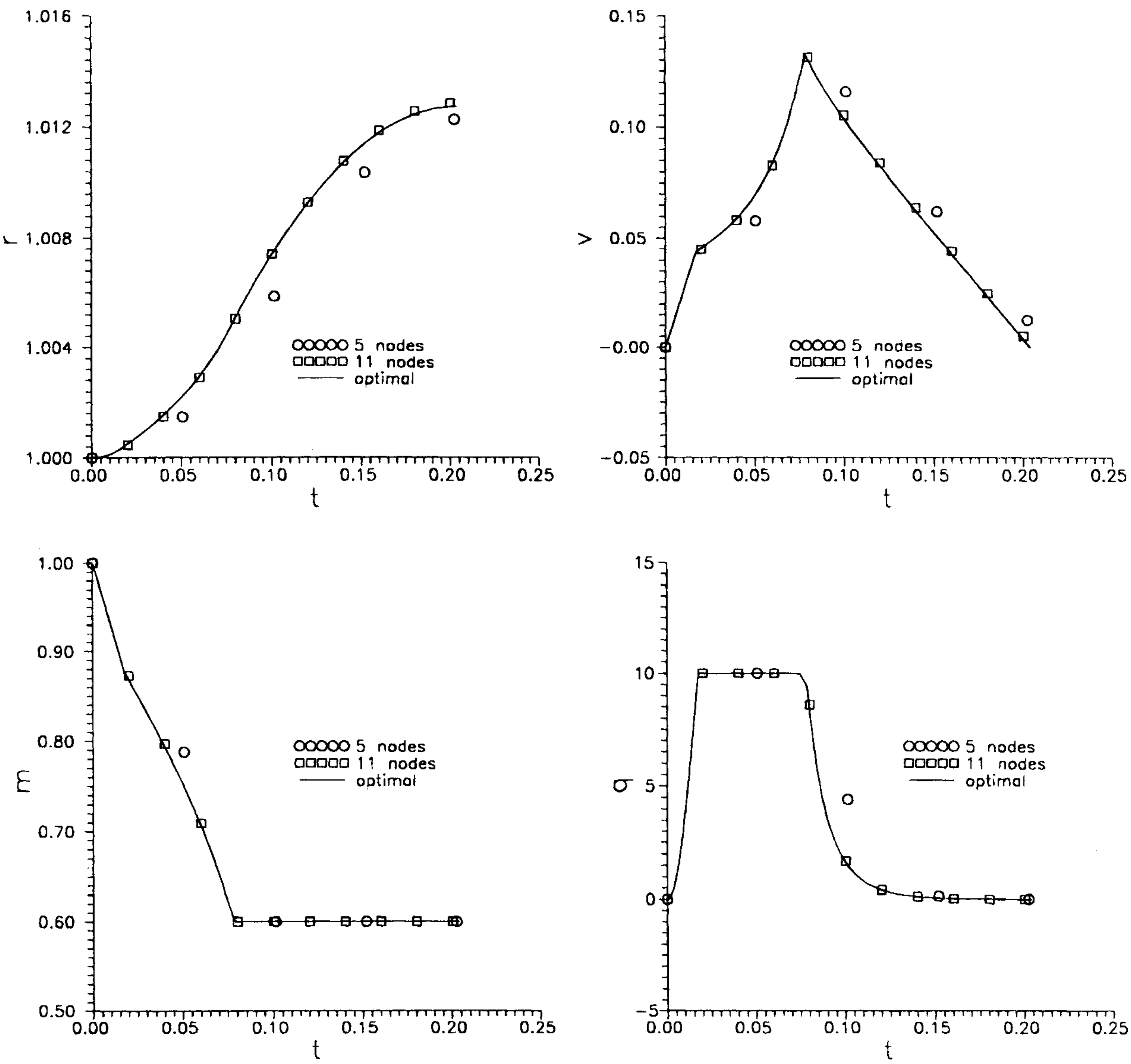

Fig. 4 Example 2: time histories for states r, v, m, and dynamic pressure q for $q_{max} = 10$ and various node densities.

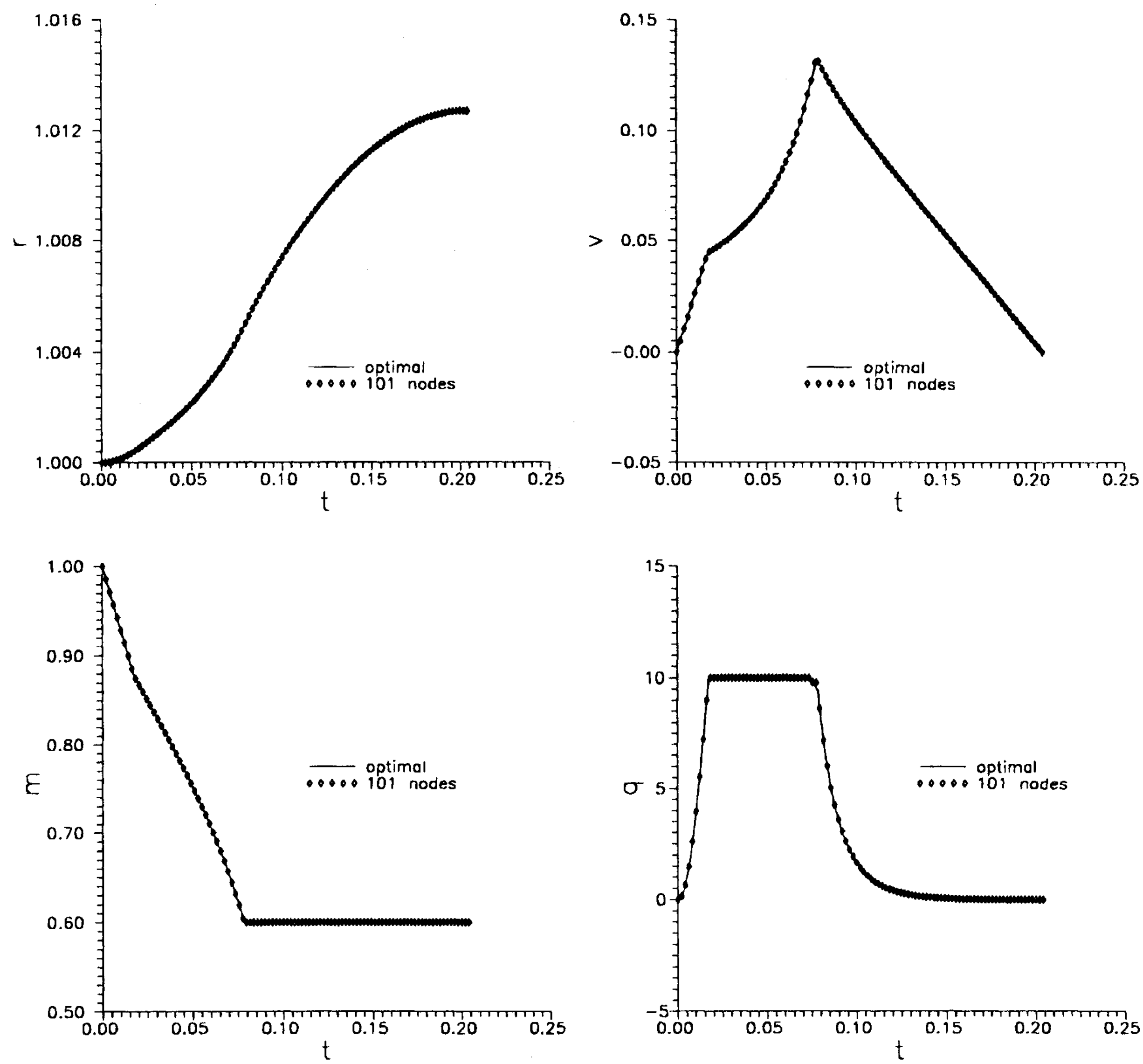

Fig. 5 Example 2: time histories for states *r*, *v*, *m*, and dynamic pressure *q* for q_{max} = 10 and nodes = 101.

subject to the constraints

$$\begin{aligned} r_0 &= 1, & v_0 &= 0 \\ m_0 &= 1, & m_N &= m_f \end{aligned} \tag{47}$$

$$\left.\begin{aligned} \dot{\bar{r}}_i - \bar{v}_i &= 0 \\ \dot{\bar{v}}_i + \frac{c \cdot \dot{\bar{m}}_i + D}{\bar{m}_i} + \frac{1}{\bar{r}_i^2} &= 0 \\ \dot{\bar{m}}_i &\le 0 \\ -\dot{\bar{m}}_i - \frac{T_{max}}{c} &\le 0 \end{aligned}\right\} \quad i = 0, \cdots, N-1 \tag{48}$$

and

$$\bar{v}_i - \sqrt{\frac{2q_{max}}{\rho_0 e^{\beta(1-r_i)}}} \le 0, \qquad i = 0, \cdots, N \tag{49}$$

Here, for all states $x \in \{r, v, m\}$, the quantities $\bar{x}_i, \dot{\bar{x}}_i$, $i = 0, \ldots, N-1$ are defined by

$$\bar{x}_i = \frac{x_{i+1} + x_i}{2}, \qquad \dot{\bar{x}}_i = \frac{x_{i+1} - x_i}{\Delta t} \tag{50}$$

with

$$\Delta t = t_f / N \tag{51}$$

and give approximate values for states and state rates in between nodes. Conditions (47) represent the initial/final conditions (42) in the discretized form, conditions (48) replace the description (40) and (41) of the underlying dynamical system, and conditions (49) enforce the state inequality constraint (43). Note that in contrast to the example problem 1, the final time t_f is free here and appears as an additional parameter in Eqs. (46–51).

A first numerical solution is generated for the simple case $N = 2$. Using the rough initial guesses

$$\begin{aligned} (r_0 \quad r_1 \quad r_2) &= (1 \quad 1 \quad 1) \\ (v_0 \quad v_1 \quad v_2) &= (0 \quad 0 \quad 0) \\ (m_0 \quad m_1 \quad m_2) &= (1 \quad 0.8 \quad 0.6) \\ t_2 &= 0.1 \end{aligned}$$

the nonlinear programming problem (46–49) converges after less than 10 iterations with the code NPSOL.[12] Initial guesses for cases with $N > 2$ were generated by linearly interpolating the results obtained with $N = 2$. For $q_{max} = \infty$, Fig. 3 shows the states r, v, m, and the dynamic pressure q vs time t, respectively. Figures 4 and 5 show an active state constraint case with $q_{max} = 10$.

It is important to note that the singular arc nature of the underlying optimal solution did not seem to have a negative effect on the convergence behavior of the differential inclusion approach. Loosely speaking, the singular nature of the optimal solution reflects the fact that along certain subarc(s) the cost criterion is only affected to second order by variations in the control. Usually, this situation negatively affects the convergence behavior of direct optimization techniques (see, for instance, Ref. 15). The differential inclusion approach seems to overcome this difficulty by avoiding an explicit parameterization of the control function of time.

Transformation of General Dynamical Systems into the Differential Inclusion Format

The method introduced in the present paper relies on a representation of the underlying dynamical system in terms of a differential inclusion format, where the set of admissible state rates is specified by a set of equality and inequality constraints on the current states and state rates. This method of modeling dynamical systems is equivalent to the usual differential equations approach, where the admissible state rates are parameterized by control variables. Even though these two approaches are equivalent, the differential equations representation is much more common and in use, so that, in practice, it is necessary to transform a given problem from the differential equations format into the differential inclusion format. In the examples presented in the paper this step is performed in an ad hoc fashion without providing a general procedure that works for all problems. To exploit the superior convergence properties in an off-the-shelf turn-key software package the development of an automatic transformation procedure is essential, especially for cases where the control elimination is not possible through noniterative analytical methods. Ongoing research efforts into this direction are beyond the scope of the present paper.

Summary and Conclusions

A method for generating approximate solutions to optimal control problems has been introduced in this paper. By employing the concepts of attainable sets and differential inclusions, a numerical representation of the dynamical system is achieved that is completely void of controls. This leads to a discretized problem formulation of relatively low dimensionality. The absence of fast "moving" controls also improves the convergence properties and enhances robustness.

The new method is illustrated on a detailed treatment of a simple double integrator problem. The performance of the algorithm is demonstrated on a one-dimensional rocket ascent problem (Goddard problem) with and without an active dynamic pressure limit.

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